Solutions of tetrahedron and 3D reflection equations from quantum cluster algebras

Atsuo Kuniba (Univ. Tokyo)

Program on Classical, Quantum and Probabilistic Integrable Systems
- Novel Interactions and Applications @ CMSA, Harvard, Boston

31 March 2025

0. **Integrability in 2D (prologue)**

1. **Tetrahedron and 3D reflection equations**

2. **A new solution**

3. **Derivation from quantum cluster algebra**

4. **Tetrahedron equality as duality**

5. **Outlook**

References

R. Inoue, A.K, Y. Terashima,

Quantum cluster algebras and 3D integrability: Tetrahedron and 3D reflection equations.

IMRN(2024) math.QA 2310.14493 Fock-Goncharov quiver (Today's talk mainly)

Tetrahedron equation and quantum cluster algebras

JPA(2024) math.QA 2310.14529 Square quiver

R.I, A.K, Xiaoyue Sun, Y.T, Junya Yagi

Solutions of tetrahedron equation from quantum cluster algebra associated with symmetric butterfly quiver

SIGMA(2024) math.QA 2403.08814 Symmetric butterfly quiver (covering/unifying many known solutions)

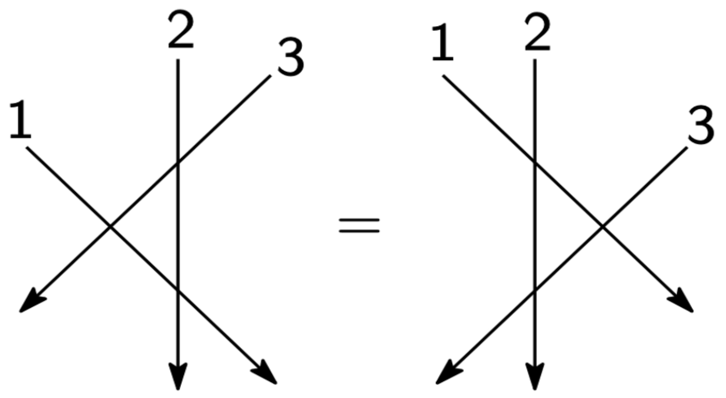
0. Integrability in 2D

Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \ \in \mathrm{End}(V^{\otimes 3}),$$

where R_{ij} acts on the ith and jth components:

$$R_{12} : \color{red}{V} \otimes \color{red}{V} \otimes V, \quad R_{23} : V \otimes \color{red}{V} \otimes \color{red}{V}, \quad R_{13} : \color{red}{V} \otimes V \otimes \color{red}{V}$$



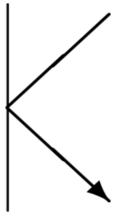
Braid Move

Wiring diagram

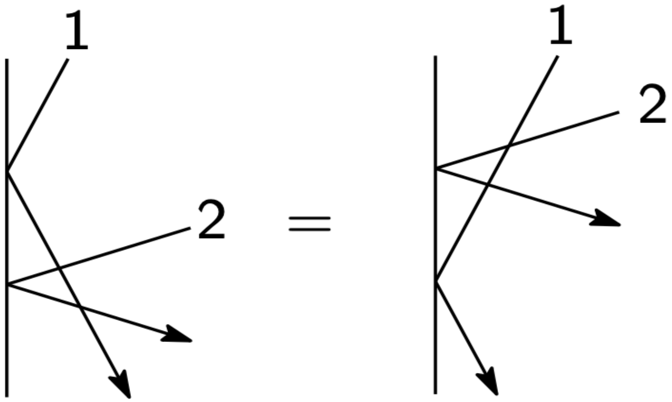
- Factorization of 3 particle scattering amplitude into 2 body ones
- Commutativity of row transfer matrices in lattice models

Key to quantum integrability in 2D

Integrability in the presence of boundary reflections

$K =$  $: V \to V$ (reflection amplitude matrix)

Reflection equation



Reflection move

Wiring diagram

$$R_{21}K_2R_{12}K_1 = K_1R_{21}K_2R_{12} \in \mathrm{End}(V^{\otimes 2})$$

$$(K_1 = K \otimes 1, \quad K_2 = 1 \otimes K)$$

$\cdots$ Factorization condition at the boundary

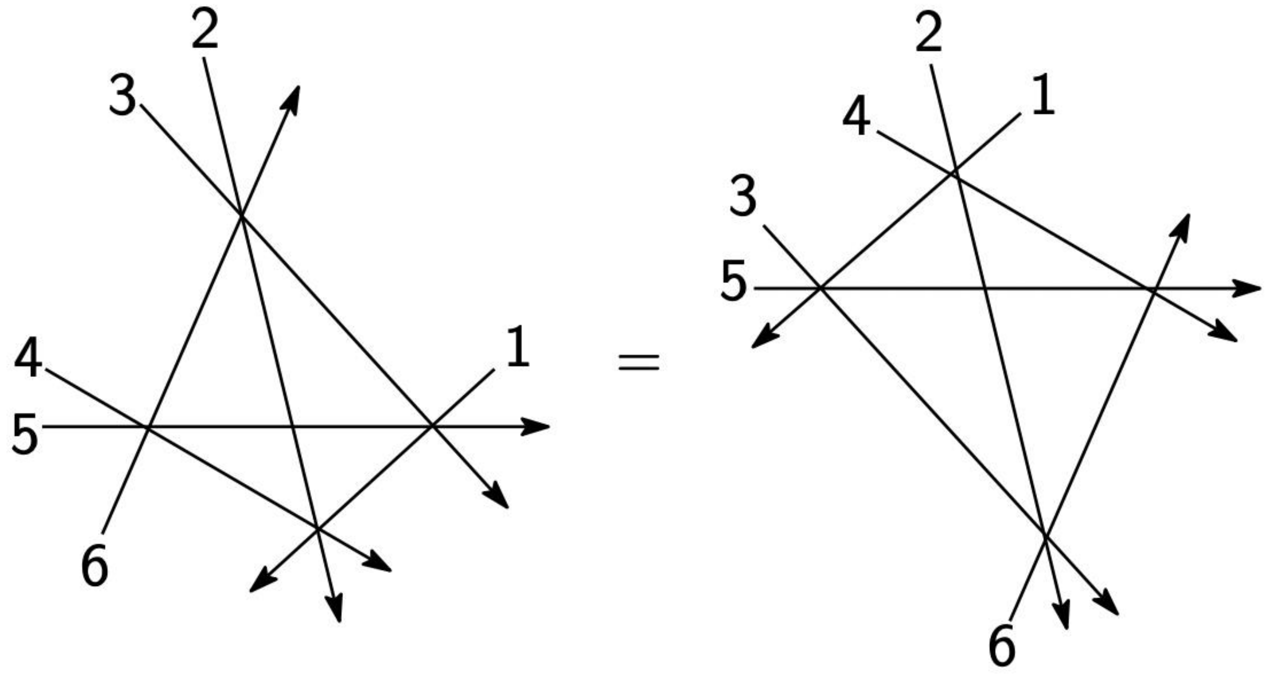
1. Tetrahedron and 3D reflection equations

(3D analogue of the Yang-Baxter and reflection eqs.)

Tetrahedron eq. [A.B. Zamolodchikov 80]

$$R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{135}R_{124} \ \text{ on } V^{\otimes 6}$$

$$R_{ijk} \in \mathrm{End}(\overset{i}{V} \otimes \overset{j}{V} \otimes \overset{k}{V})$$



R = local Boltzmann weights of a vertex in 3D

1. Tetrahedron and 3D reflection equations

(3D analogue of the Yang-Baxter and reflection eqs.)

Tetrahedron eq. [A.B. Zamolodchikov 80]

$$R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{135}R_{124} \ \text{ on } V^{\otimes 6}$$

$$R_{ijk} \in \mathrm{End}(\overset{i}{V} \otimes \overset{j}{V} \otimes \overset{k}{V})$$

3D reflection eq. [Isaev-Kulish 97]

$$R_{689}K_{3579}R_{249}R_{258}K_{1478}K_{1236}R_{456} = R_{456}K_{1236}K_{1478}R_{258}R_{249}K_{3579}R_{689}$$

$$\text{on } W \otimes V \otimes W \otimes V \otimes V \otimes V \otimes W \otimes V \otimes V$$

$$K_{ijkl} \in \mathrm{End}(\overset{i}{W} \otimes \overset{j}{V} \otimes \overset{k}{W} \otimes \overset{j}{V})$$

" Three upright open books on a desk with their spines undergoing a Yang-Baxter move."

1. Tetrahedron and 3D reflection equations

(3D analogue of the Yang-Baxter and reflection eqs.)

Tetrahedron eq. [A.B. Zamolodchikov 80]

$$R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{135}R_{124} \text{ on } V^{\otimes 6}$$

$$R_{ijk} \in \mathrm{End}(\overset{i}{V} \otimes \overset{j}{V} \otimes \overset{k}{V})$$

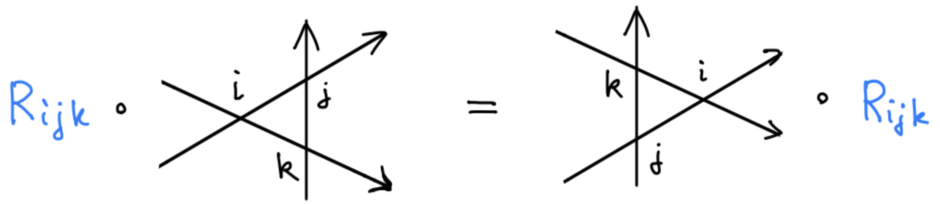
3D reflection eq. [Isaev-Kulish 97]

$$R_{689}K_{3579}R_{249}R_{258}K_{1478}K_{1236}R_{456} = R_{456}K_{1236}K_{1478}R_{258}R_{249}K_{3579}R_{689}$$

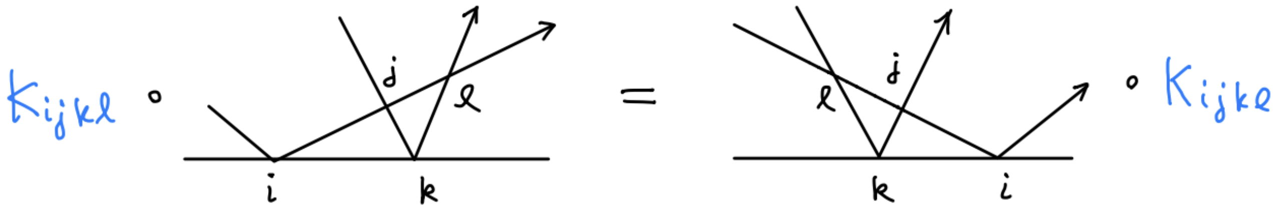
$$\text{on } W \otimes V \otimes W \otimes V \otimes V \otimes V \otimes W \otimes V \otimes V$$

$$K_{ijkl} \in \mathrm{End}(\overset{i}{W} \otimes \overset{j}{V} \otimes \overset{k}{W} \otimes \overset{j}{V})$$

They are compatibility conditions of the quantized Yang-Baxter eq. and quantized reflection eq., which are the *usual* Yang-Baxter and reflection equations up to conjugation.

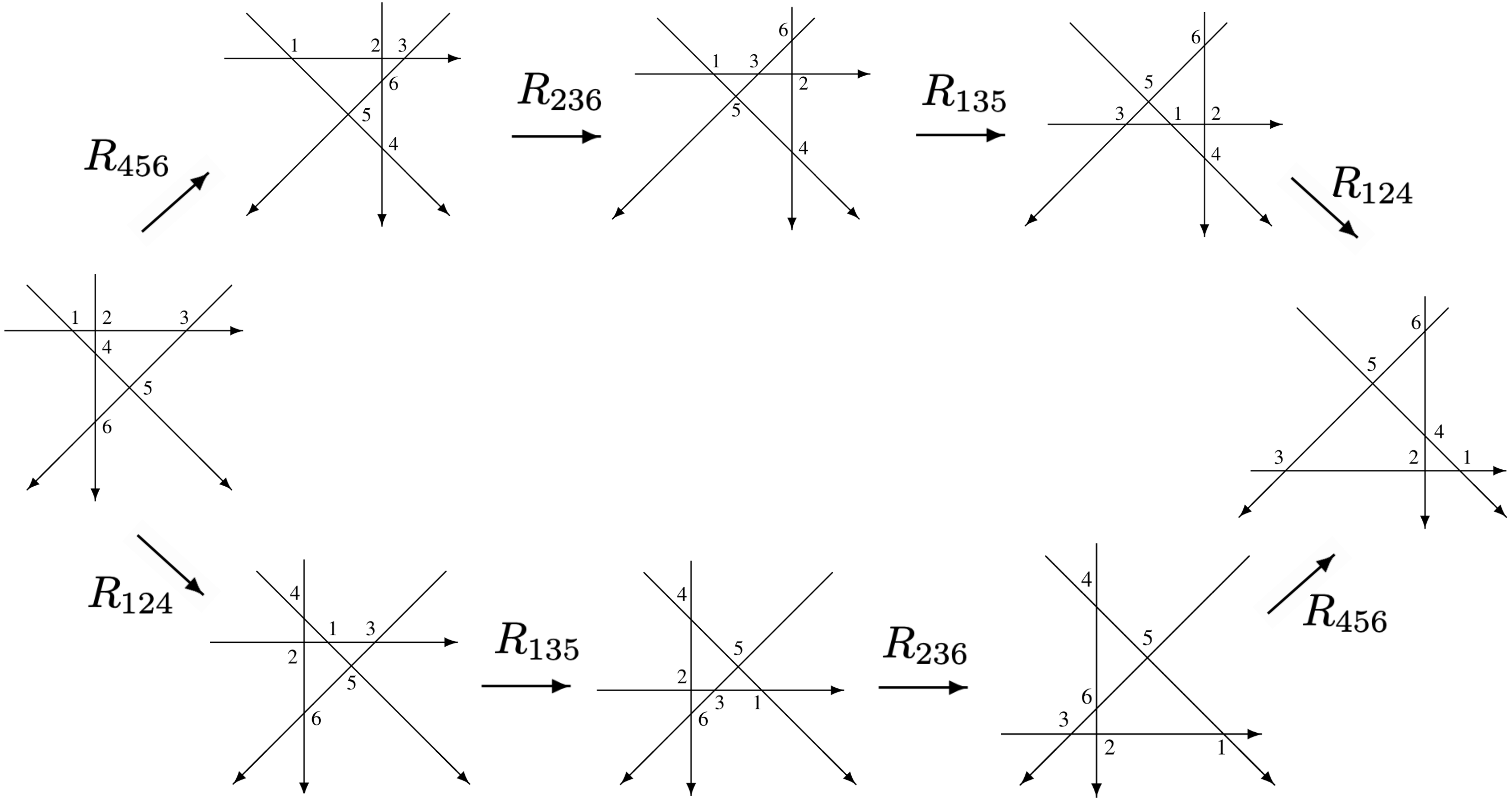


Braid move



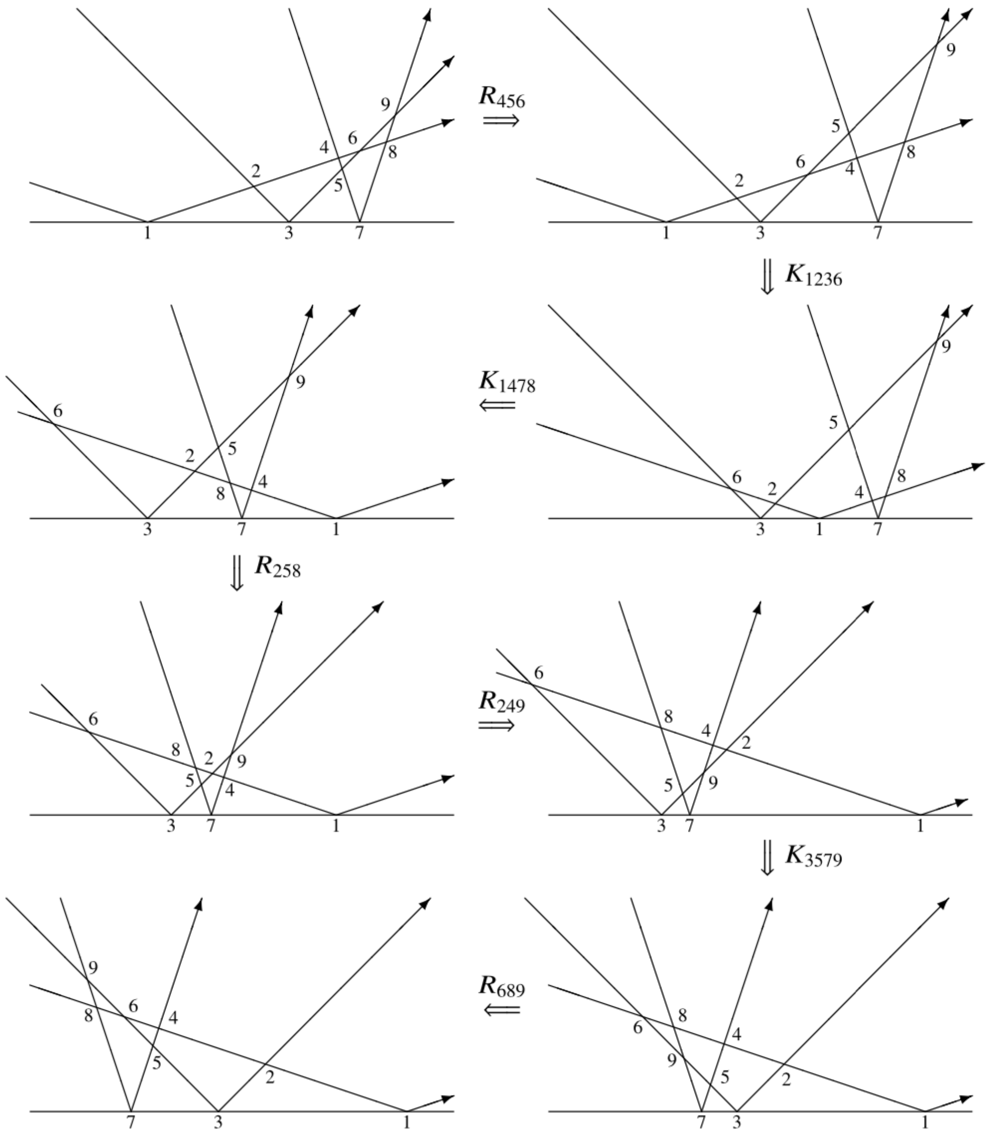
Reflection move

Now that R and K play the role of *structure constants*, they have to satisfy the compatibility condition under introducing one more arrow:

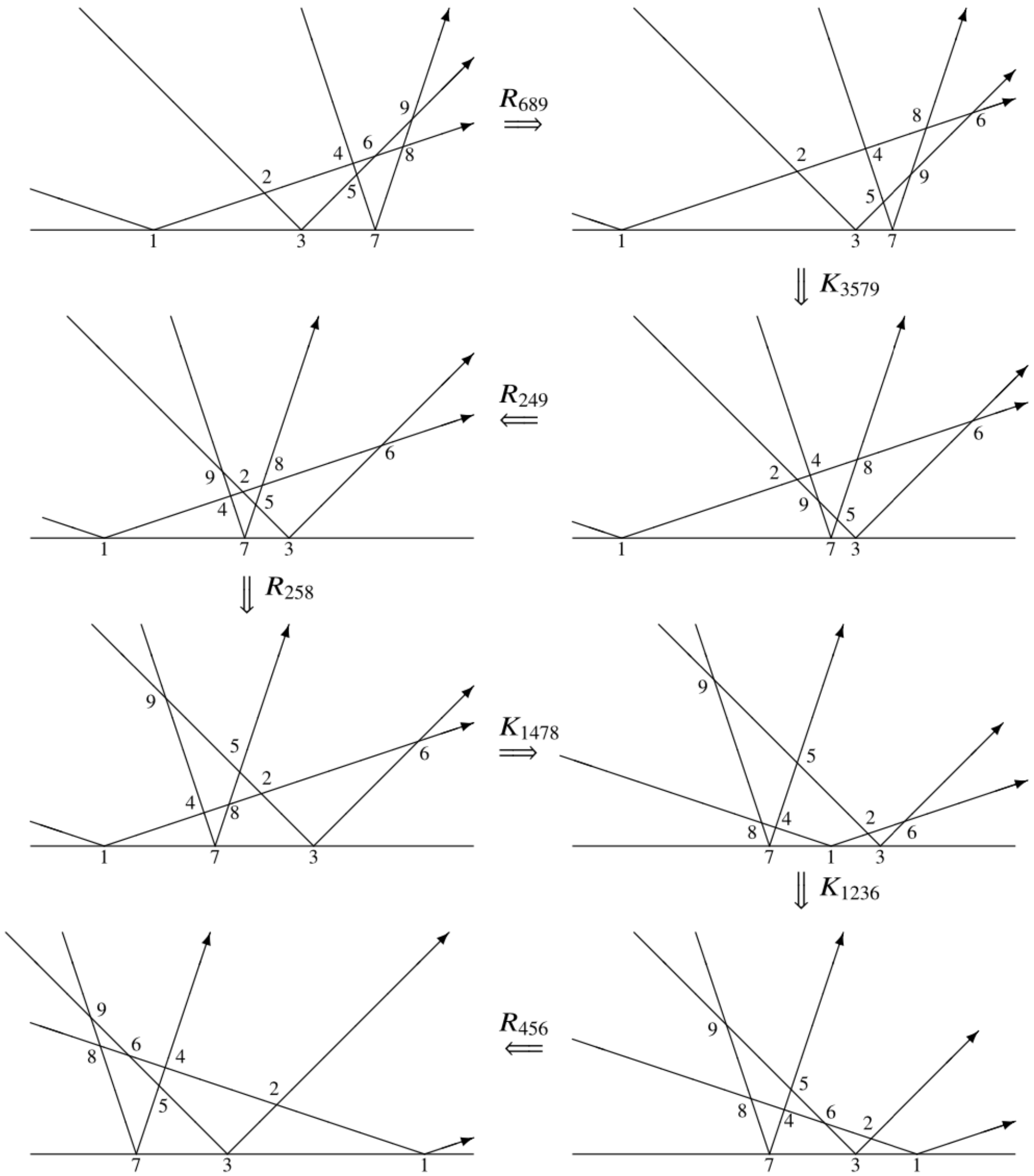


$$R_{689}K_{3579}R_{249}R_{258}K_{1478}K_{1236}R_{456} = R_{456}K_{1236}K_{1478}R_{258}R_{249}K_{3579}R_{689}$$

LHS



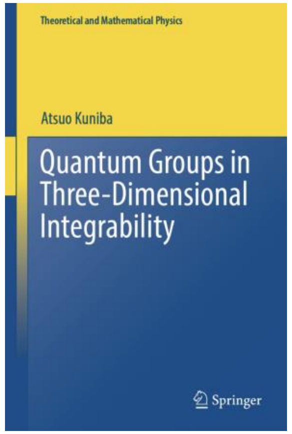
RHS



Several interesting solutions are known for the tetrahedron equation by Zamolodchikov, Baxter, Kapranov-Voevodsky, Bazhanov, Mangazeev, Sergeev, Stroganov, ...

Only a few solutions are known for the 3D reflection equation by K-Okado, Yoneyama (as of 2022).

There are quantum group theoretical approaches based on quantized coordinate rings by [Kapranov-Voevodsky 94] and PBW basis of U_q^+ by [Sergeev 08].



They are equivalent beyond type A [K-Okado-Yamada 13] and have been developed extensively with many applications.

The aim of this talk is to develop another approach [Sun-Yagi 22], where these diagrams are complemented by quivers that facilitate the efficient operation of quantum cluster algebras.

We focus on the Fock-Goncharov quivers, devise a new realization of quantum Y-variables using q-Weyl algebras, and obtain a new solution.

2. New solution (emerging from quantum cluster algebra associated with the Fock-Goncharov quiver)

$$\mathcal{R}_{ijk} = \Psi_q(e^{p_i+u_i+p_k-u_k-p_j+\lambda_{ik}})\rho_{jk}\, e^{\frac{1}{\hbar}p_i(u_k-u_j)}e^{\frac{\lambda_{jk}}{\hbar}(u_k-u_i)},$$

$$\mathcal{K}_{ijkl} = \Psi_{q^2}(e^{p_j+u_j+p_l-u_l-2p_k+\lambda_{jl}})\Psi_q(e^{p_i+u_i+p_k-u_k-p_j+\lambda_{ik}})\Psi_{q^2}(e^{p_j+u_j+p_l-u_l-2p_k+\lambda_{jl}})^{-1}$$
$$\times\, \rho_{jl}\, e^{\frac{1}{\hbar}p_i(u_l-u_j)}e^{\frac{\lambda_{jl}}{2\hbar}(2u_k-2u_i+u_l-u_j)}.$$

$$\Psi_q(X) = \frac{1}{(-qX;q^2)_\infty} : \ \text{quantum dilogarithm} \qquad (z;q)_m = (1-z)(1-qz)\cdots(1-zq^{m-1})$$

Key properties

$$\Psi_q(q^2U)\Psi_q(U)^{-1} = 1+qU,$$
$$\Psi_q(U)\Psi_q(W) = \Psi_q(W)\Psi_q(q^{-1}UW)\Psi_q(U) \ \text{ if } UW = q^2WU$$

(pentagon identity)

$$[p_i,u_j] = \begin{cases} 2\delta_{ij}\hbar & i,j\in\{3,6,9\} \\ \delta_{ij}\hbar & \text{otherwise}\end{cases} \left(\begin{array}{c}[p_i,u_j]=\delta_{ij}\hbar \\ \text{for tetrahedron eq.}\end{array}\right) \ [p_i,p_j]=[u_i,u_j]=0 \ : \ \text{canonical variables}$$

$$\rho_{ij} = \text{transposition} \ \ p_i \leftrightarrow p_j,\ u_i \leftrightarrow u_j \qquad q = e^{\hbar}, \ \ \lambda_{ij} = \lambda_i - \lambda_j$$

3. Derivation from quantum cluster algebra (Fock-Goncharov(09) q-deforming Fomin-Zelevinsky(07))

$B \leftrightarrow Q$: quiver with vertices $1, \ldots, n$

$\mathrm{Seed} = (B, \mathbf{Y})$

$B = (b_{ij})_{i,j=1}^{n}, \;\; b_{ij} = -b_{ji} \in \mathbb{Z}/2$: Exchange matrix (Type A only)

$b_{ij} = 1$

$i \longrightarrow j$

$\mathbf{Y} = (Y_1, \ldots, Y_n), \quad Y_i Y_j = q^{2b_{ij}} Y_j Y_i$: Y-variables

$b_{ij} = 1/2$

$i \dashrightarrow j$

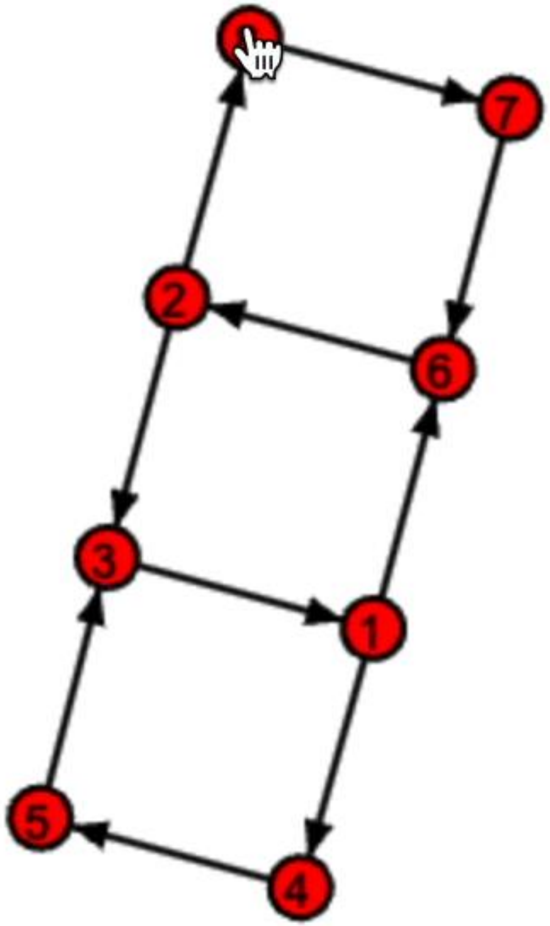
$\mathbb{F}(\mathbf{Y}) = \mathbb{F}(B, \mathbf{Y})$: non-commutative fraction field generated by $\mathbf{Y}$

Mutation

$$\mu_k(B, \mathbf{Y}) = (B', \mathbf{Y}') \qquad k \in \{1, \ldots, n\}$$

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + [b_{ki}]_+ b_{kj} + [b_{kj}]_+ b_{ik} & \text{otherwise} \end{cases} \qquad [x]_+ = \max(x, 0)$$

$$Y'_i = \begin{cases} Y_k^{-1} & i = k \\ q^{b_{ik}[b_{ki}]_+} Y_i Y_k^{[b_{ki}]_+} \prod_{m=1}^{|b_{ki}|} \left(1 + q^{-\mathrm{sgn}(b_{ki})(2m-1)} Y_k\right)^{-\mathrm{sgn}(b_{ki})} & i \neq k \end{cases}$$


7
2
6
3
1
5
4

μ_k on $\mathbf{Y}$ is decomposed into monomial part and dilog (automorphism) part in two $(+,-)$ ways so that the following diagram becomes commutative:

$$\begin{tikzcd} Y_i \in \mathbb{F}(\mathbf{Y}) \arrow[r, "\mu_k"] \arrow[d, mapsto] & \mathbb{F}(\mathbf{Y}) \\ Y_i' \in \mathbb{F}(\mathbf{Y}') \arrow[r, "\tau_{k,\pm}"', "\text{monomial part}"' yshift=-12pt] & \mathbb{F}(\mathbf{Y}) \arrow[u, "\mu^\sharp_{k,\pm}\ \text{dilog part}"'] \end{tikzcd}$$

$$\tau_{k,\varepsilon}(Y_i') = \begin{cases} Y_k^{-1} & i = k \\ q^{-b_{ik}[\varepsilon b_{ik}]_+} Y_i Y_k^{[\varepsilon b_{ik}]_+} & i \neq k \end{cases} \qquad (\varepsilon = \pm)$$

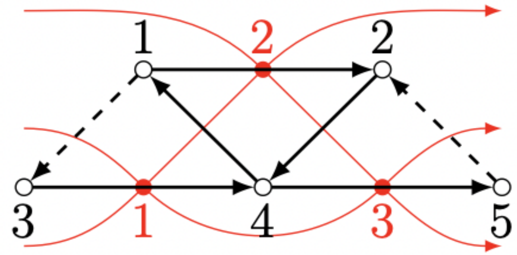
$$\mu^\sharp_{k,\varepsilon} = \mathrm{Ad}\big(\Psi_q(Y_k^\varepsilon)^\varepsilon\big), \quad \text{i.e.} \quad \mu^\sharp_{k,\varepsilon}(Y_i) = \Psi_q(Y_k^\varepsilon)^\varepsilon Y_i \Psi_q(Y_k^\varepsilon)^{-\varepsilon}$$

Compositions of $\mu_k^* := \mathrm{Ad}\big(\Psi_q(Y_k^\varepsilon)^\varepsilon\big)\tau_{k,\varepsilon} : \mathbb{F}(\mathbf{Y}') \to \mathbb{F}(\mathbf{Y})$ are called cluster transformations.

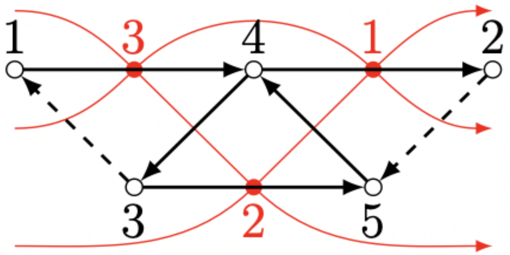
Example $\quad \underset{Y_1}{\overset{1}{\circ}} \longrightarrow \underset{Y_2}{\overset{2}{\circ}} \quad \xrightarrow{\mu_2} \quad \underset{Y_1(1+qY_2^{-1})^{-1}}{\overset{1}{\circ}} \longleftarrow \underset{Y_2^{-1}}{\overset{2}{\circ}} \qquad b_{12} = 1 = -b_{21},\ Y_1Y_2 = q^2Y_2Y_1$

$$\begin{tikzcd}[column sep=small] & q^{-1}Y_1Y_2 \arrow[r, "\mu^\sharp_{2,+}"] & q^{-1}Y_1\Psi_q(q^{-2}Y_2)\Psi_q(Y_2)^{-1}Y_2 = q^{-1}Y_1(1+q^{-1}Y_2)^{-1}Y_2 \arrow[dr, equal] & \\ Y_1' \arrow[ur, "\tau_{2,+}"] \arrow[dr, "\tau_{2,-}"'] & & & Y_1(1+qY_2^{-1})^{-1} \\ & Y_1 \arrow[r, "\mu^\sharp_{2,-}"] & \Psi_q(Y_2^{-1})^{-1}Y_1\Psi_q(Y_2^{-1}) = Y_1\Psi_q(q^2Y_2^{-1})^{-1}\Psi_q(Y_2^{-1}) \arrow[ur, equal] & \end{tikzcd}$$

Wiring diagrams (red) and the Fock-Goncharov (FG) quivers (black): Type A_2



$$\xrightarrow[R_{123}]{\mu_4}$$



$$(Y_1,\ldots,Y_5) \xleftarrow{\mu_4^*} (Y_1',\ldots,Y_5')$$

FG quiver $\cong$ dual of wiring diagram

FG quivers are designed in such a way that the braid move R_{123} and the mutation μ_4 are compatible.

$$\mu_4^* : \begin{pmatrix} Y_1' \\ Y_2' \\ Y_3' \\ Y_4' \\ Y_5' \end{pmatrix} \overset{\tau_{4,+}}{\mapsto} \begin{pmatrix} Y_1 \\ q^{-1}Y_2Y_4 \\ q^{-1}Y_3Y_4 \\ Y_4^{-1} \\ Y_5 \end{pmatrix} \overset{\mathrm{Ad}(\Psi_q(Y_4))}{\mapsto} \begin{pmatrix} Y_1(1+qY_4) \\ Y_2(1+qY_4^{-1})^{-1} \\ Y_3(1+qY_4^{-1})^{-1} \\ Y_4^{-1} \\ Y_5(1+qY_4) \end{pmatrix}$$

Associated cluster transformation

The transformation R_{123} of the wiring diagram satisfies the tetrahedron equation (as noted earlier)

$$R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{135}R_{124}$$

Key idea: Upgrade it into an equality of cluster transformations

$A_2 \hookrightarrow A_3$

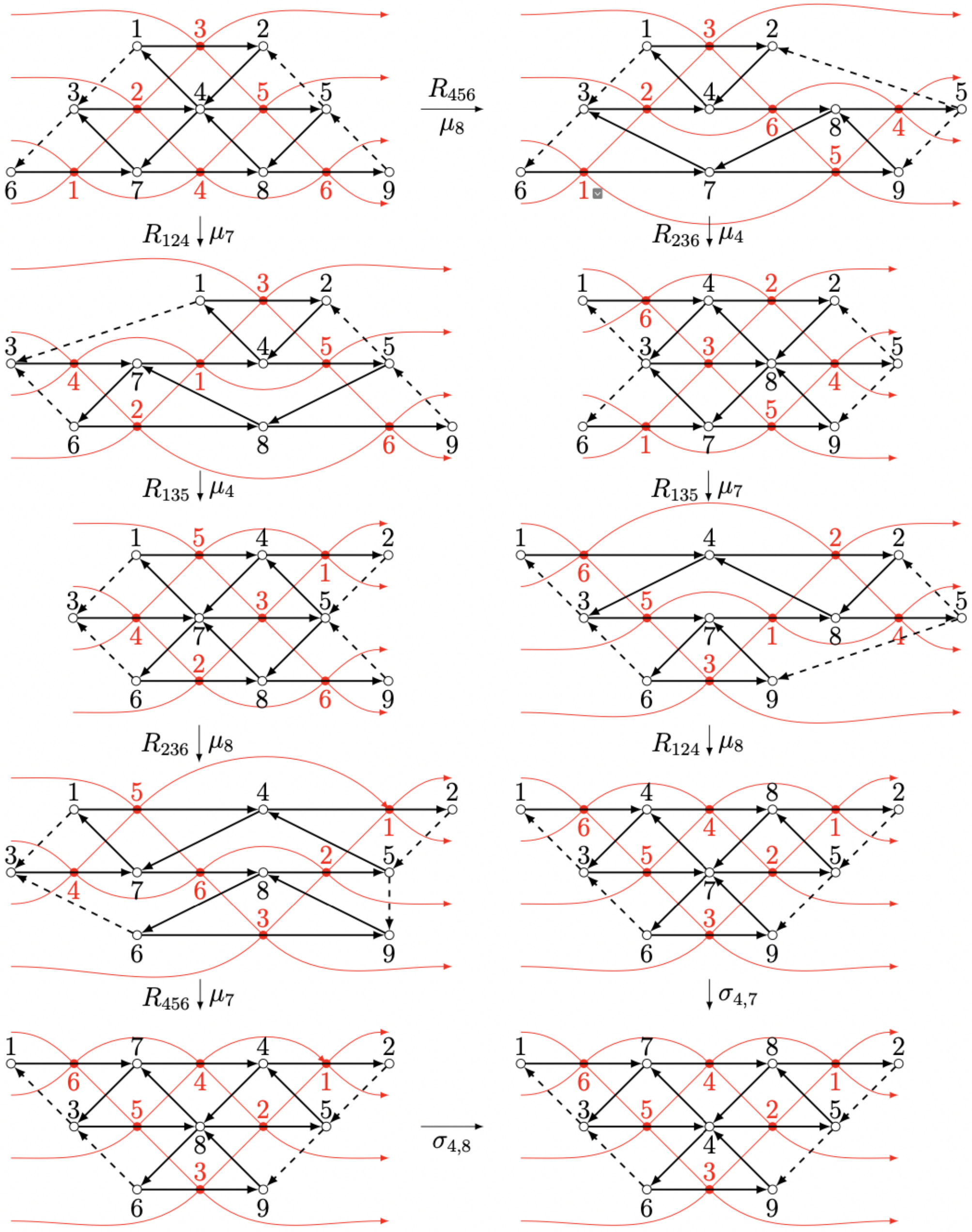
Wiring diagrams (red) which are successively transformed by braid moves denoted by R_{ijk}

The figure shows that R_{ijk} satisfies the tetrahedron equation (as noted before).

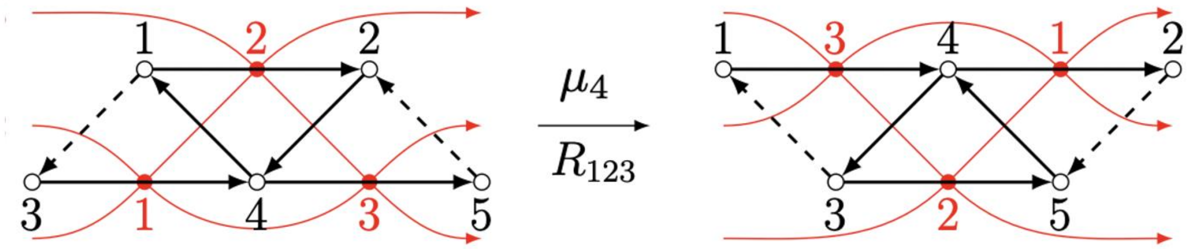
They are associated with the FG quivers (black) which are transformed by mutations μ_r

Quantum cluster algebra ensures the equality of the corresponding cluster transformations!

Our solution is extracted as an operator whose adjoint induces the cluster transformation corresponding to R_{ijk}



Embedding into q-Weyl algebras



$$(Y_1, \dots, Y_5) \xleftarrow{\mu_4^*} (Y_1', \dots, Y_5')$$

$$\begin{aligned} Y_1 Y_2 &= q^2 Y_2 Y_1 \\ Y_1 Y_3 &= q Y_3 Y_1 \\ Y_1 Y_4 &= q^{-2} Y_4 Y_1 \\ Y_1 Y_5 &= Y_5 Y_1, \ \text{etc} \end{aligned} \qquad \begin{aligned} Y_1' Y_2' &= Y_2' Y_1' \\ Y_1' Y_3' &= q^{-1} Y_3' Y_1' \\ Y_1' Y_4' &= q^2 Y_4' Y_1' \\ Y_1' Y_5' &= Y_5' Y_1', \ \text{etc} \end{aligned}$$

canonical commutation relations

The q-commutativity becomes automatic in the following parameterization using q-Weyl algebra

Introduce canonical variables:

$$[p_i, u_j] = \hbar \delta_{ij}, \quad [p_i, p_j] = [u_i, u_j] = 0$$

$e^{\pm p_i}, e^{\pm u_i}$ are generators of q-Weyl algebra

with the relation $e^{p_i} e^{u_j} = q^{\delta_{ij}} e^{u_j} e^{p_i}$

$$(\, q = e^{\hbar}, \quad \kappa_j = e^{\lambda_j}, \quad \lambda_{ij} = \lambda_i - \lambda_j \,)$$

$$\begin{aligned} Y_1 &= \kappa_2^{-1} e^{p_2 - u_2 - p_1} \\ Y_2 &= \kappa_2 e^{p_2 + u_2 - p_3} \\ Y_3 &= \kappa_1^{-1} e^{p_1 - u_1} \\ Y_4 &= \kappa_1 \kappa_3^{-1} e^{p_1 + u_1 + p_3 - u_3 - p_2} \\ Y_5 &= \kappa_3 e^{p_3 + u_3} \end{aligned} \qquad \begin{aligned} Y_1' &= \kappa_3^{-1} e^{p_3 - u_3} \\ Y_2' &= \kappa_1 e^{p_1 + u_1} \\ Y_3' &= \kappa_2^{-1} e^{p_2 - u_2 - p_3} \\ Y_4' &= \kappa_1^{-1} \kappa_3 e^{p_3 + u_3 + p_1 - u_1 - p_2} \\ Y_5' &= \kappa_2 e^{p_2 + u_2 - p_1} \end{aligned}$$

Moreover, what is no less remarkable in the q-Weyl algebra parameterization is that not only the dilogarithm part, but also the monomial part

$$\begin{pmatrix} Y_1' \\ Y_2' \\ Y_3' \\ Y_4' \\ Y_5' \end{pmatrix} \overset{\tau_{4,+}}{\mapsto} \begin{pmatrix} Y_1 \\ q^{-1}Y_2Y_4 \\ q^{-1}Y_3Y_4 \\ Y_4^{-1} \\ Y_5 \end{pmatrix}$$

is realized entirely as an adjoint as

$$\tau_{4,+} = \mathrm{Ad}(P_{123}) \qquad\qquad P_{123} = \rho_{23}\, e^{\frac{1}{\hbar}p_1(u_3-u_2)} e^{\frac{\lambda_{23}}{\hbar}(u_3-u_1)}$$

Example

$$\begin{aligned} \mathrm{Ad}(P_{123})(e^{p_3}) &= \rho_{23}\, e^{\frac{1}{\hbar}p_1(u_3-u_2)} \underline{e^{\frac{\lambda_{23}}{\hbar}(u_3-u_1)} e^{p_3} e^{-\frac{\lambda_{23}}{\hbar}(u_3-u_1)}} e^{-\frac{1}{\hbar}p_1(u_3-u_2)} \rho_{23} \\ &= \rho_{23}\, \underline{e^{\frac{1}{\hbar}p_1(u_3-u_2)} e^{-\lambda_{23}}} \underline{e^{p_3} e^{-\frac{1}{\hbar}p_1(u_3-u_2)}} \rho_{23} \\ &= \rho_{23}\, e^{-p_1-\lambda_{23}} e^{p_3} \rho_{23} = e^{p_2-p_1-\lambda_{23}}. \end{aligned}$$

Underlined parts are treated by the Baker-Campbell-Hausdorff formula

Therefore, the cluster transformation μ_4^* becomes totally an adjoint as

$$\mu_4^* = \mathrm{Ad}(\Psi_q(Y_4))\tau_{4,+} = \mathrm{Ad}(\Psi_q(Y_4))\mathrm{Ad}(P_{123}) = \mathrm{Ad}\Big(\underbrace{\Psi_q(Y_4)P_{123}}_{\mathcal{R}_{123}}\Big)$$

$$\mathcal{R}_{123} = \Psi_q(Y_4)P_{123} = \Psi_q(e^{p_1+u_1+p_3-u_3-p_2+\lambda_{13}})\rho_{23}e^{\frac{1}{\hbar}p_1(u_3-u_2)}e^{\frac{\lambda_{23}}{\hbar}(u_3-u_1)}$$

$$= \mathcal{R}(\lambda_1,\lambda_2,\lambda_3)_{123}$$

Theorem. The tetrahedron equation with spectral parameters is valid:

$$\mathcal{R}(\lambda_4,\lambda_5,\lambda_6)_{456}\mathcal{R}(\lambda_2,\lambda_3,\lambda_6)_{236}\mathcal{R}(\lambda_1,\lambda_3,\lambda_5)_{135}\mathcal{R}(\lambda_1,\lambda_2,\lambda_4)_{124}$$
$$= \mathcal{R}(\lambda_1,\lambda_2,\lambda_4)_{124}\mathcal{R}(\lambda_1,\lambda_3,\lambda_5)_{135}\mathcal{R}(\lambda_2,\lambda_3,\lambda_6)_{236}\mathcal{R}(\lambda_4,\lambda_5,\lambda_6)_{456}$$

Outline so far

Basic observation (not so new)

Braid moves of wiring diagrams satisfy the tetrahedron equation.

First key step

Associating FG quivers to the wiring diagrams, it can be upgraded to an equality of cluster transformations, which is a rational transformations of q-commuting Y variables.

Second key step

Embedding into the q-Weyl algebra makes the cluster transformation into the form Ad($\mathcal{R}$)

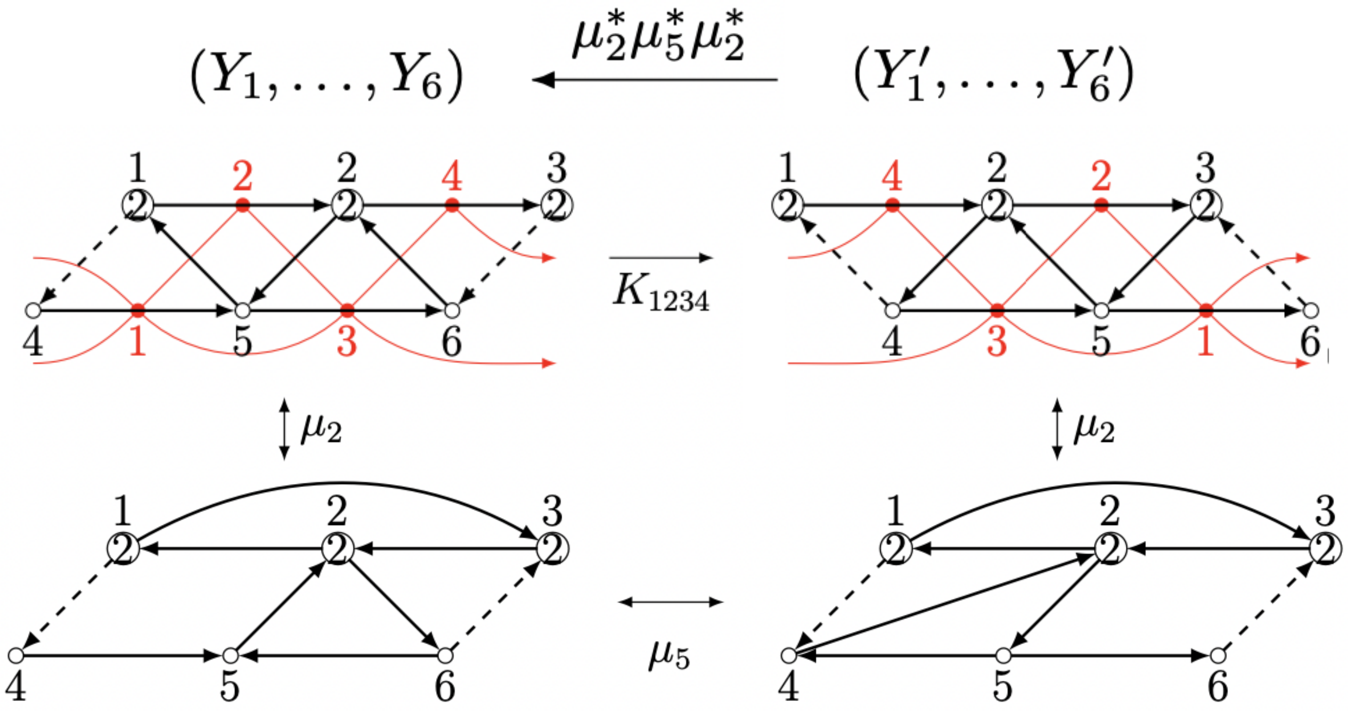
($\mathcal{R}$ = product of quantum dilogarithm and the monomial part.)

Some more work shows

$\mathcal{R}$ itself satisfies the tetrahedron equation.

Wiring diagrams (red) and the FG quivers (black) for K : Type C_2

FG quivers are *weighted*. (② = weight 2 node, Exchange matrices are only skew-*symmetrizable*)



A single reflection move corresponds to the composition of three mutations

The transformation K_{1234} of the wiring diagram induces the following cluster transformation:

$$\mu_2^* \, \mu_5^* \, \mu_2^* = \mathrm{Ad}(\Psi_{q^2}(Y_2)\Psi_q(Y_5)\Psi_{q^2}(Y_2)^{-1})\tau_{2,+}\tau_{5,+}\tau_{2,-}$$

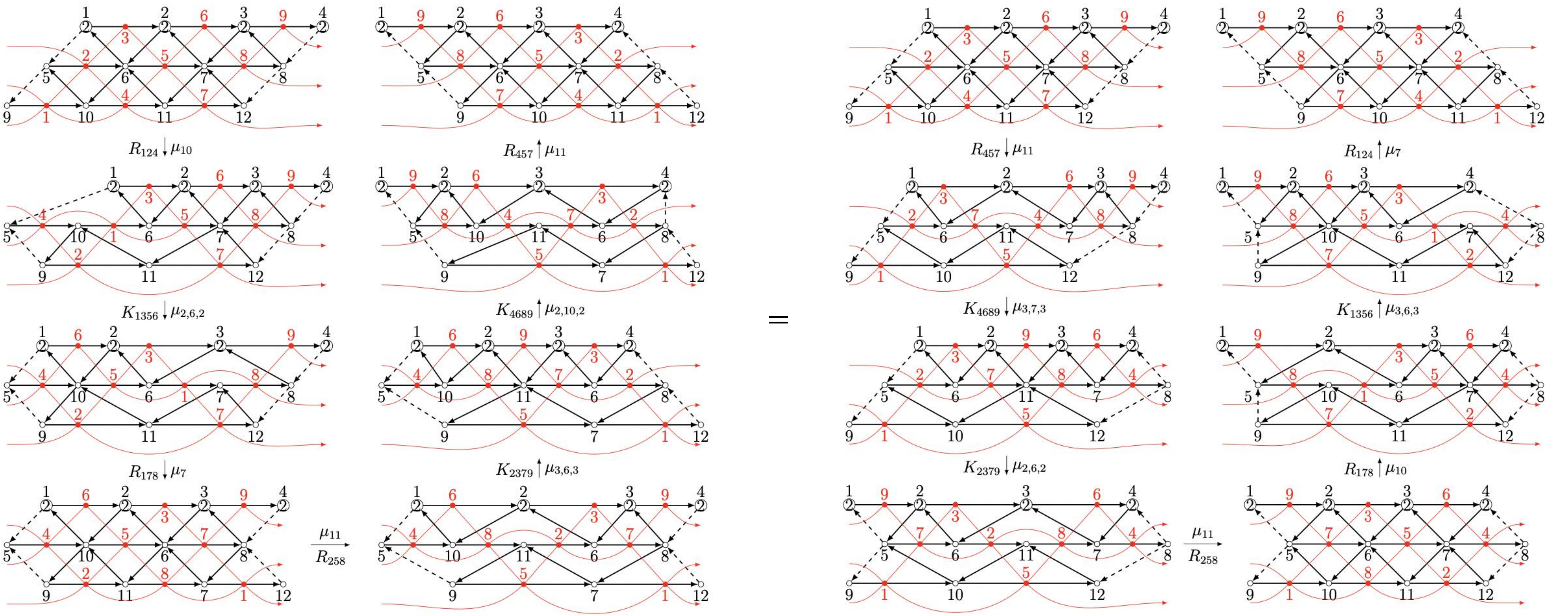
The cluster transformation induced by K_{1234}

$$\mu_2^*\mu_5^*\mu_2^* : \begin{pmatrix} Y_1' \\ Y_2' \\ Y_3' \\ Y_4' \\ Y_5' \\ Y_6' \end{pmatrix} \xmapsto{\tau_{2,+}\tau_{5,+}\tau_{2,-}} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ q^{-1}Y_4Y_5 \\ q^2Y_5^{-1}Y_2^{-1} \\ q^{-1}Y_2Y_5Y_6 \end{pmatrix} \xmapsto{\mathrm{Ad}(\Psi_{q^2}(Y_2)\Psi_q(Y_5)\Psi_{q^2}(Y_2)^{-1})} \begin{pmatrix} Y_1\Lambda_0 \\ \Lambda_1^{-1}\Lambda_2^{-1}Y_2 \\ \Lambda_0^{-1}Y_3\Lambda_1\Lambda_2 \\ q^{-1}\Lambda_0^{-1}Y_4Y_5\Lambda_1 \\ q^2Y_5^{-1}Y_2^{-1}\Lambda_0 \\ q^{-1}\Lambda_1^{-1}Y_2Y_5Y_6 \end{pmatrix}$$

$$\Lambda_0 = 1 + (q+q^3)Y_5 + q^4Y_5^2(1+q^2Y_2), \quad \Lambda_1 = 1 + qY_5(1+q^2Y_2), \quad \Lambda_2 = 1 + q^3Y_5(1+q^2Y_2)$$

Our solution (appearing after 3 pages) is an operator whose adjoint induces this rational transformation of q-commuting Y variables.

For three reflecting wires (red), there are two ways to reverse the order of reflections: $C_2 \hookrightarrow C_3$



The corresponding transformations K and R satisfy the 3D reflection equation (as noted earlier)

$$R_{457}K_{4689}K_{2379}R_{258}R_{178}K_{1356}R_{124} = R_{124}K_{1356}R_{178}R_{258}K_{2379}K_{4689}R_{457}$$

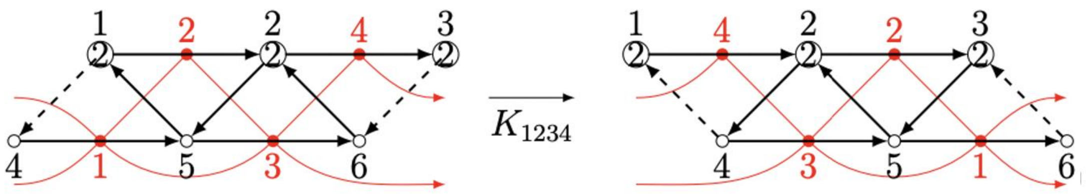
First key step:

Quantum cluster algebra ensures that the cluster transformations corresponding to the two sides coincide.

Second key step: Embedding of Y-variables into q-Weyl algebras

$$
\begin{aligned}
Y_1 &\mapsto \kappa_2^{-1} e^{p_2-u_2-2p_1},\\
Y_2 &\mapsto \kappa_2\kappa_4^{-1} e^{p_2+u_2+p_4-u_4-2p_3},\\
Y_3 &\mapsto \kappa_4\, e^{p_4+u_4},\\
Y_4 &\mapsto \kappa_1^{-1} e^{p_1-u_1},\\
Y_5 &\mapsto \kappa_1\kappa_3^{-1} e^{p_1+u_1+p_3-u_3-p_2},\\
Y_6 &\mapsto \kappa_3\, e^{p_3+u_3-p_4},
\end{aligned}
$$

$$(Y_1,\dots,Y_6) \xleftarrow{\mu_2^*\mu_5^*\mu_2^*} (Y_1',\dots,Y_6')$$



$$
\begin{aligned}
Y_1' &\mapsto \kappa_4^{-1} e^{p_4-u_4},\\
Y_2' &\mapsto \kappa_4\kappa_2^{-1} e^{p_4+u_4+p_2-u_2-2p_3},\\
Y_3' &\mapsto \kappa_2\, e^{p_2+u_2-2p_1},\\
Y_4' &\mapsto \kappa_3^{-1} e^{p_3-u_3-p_4},\\
Y_5' &\mapsto \kappa_3\kappa_1^{-1} e^{p_3+u_3+p_1-u_1-p_2},\\
Y_6' &\mapsto \kappa_1\, e^{p_1+u_1}.
\end{aligned}
$$

(p_i and u_i obey the canonical commutation relation)

The embedding makes the q-commutativity of Y_i and Y_i' variables automatic.

Under this embedding, the cluster transformation for K_{1234} becomes totally an adjoint as

$$\mu_2^* \mu_5^* \mu_2^* = \mathrm{Ad}(\mathcal{K}_{1234})$$

$$\begin{aligned}\mathcal{K}_{1234} &= \mathcal{K}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)_{1234} \\ &= \Psi_{q^2}(e^{p_2+u_2+p_4-u_4-2p_3+\lambda_{24}}) \Psi_q(e^{p_1+u_1+p_3-u_3-p_2+\lambda_{13}}) \Psi_{q^2}(e^{p_2+u_2+p_4-u_4-2p_3+\lambda_{24}})^{-1} \\ &\quad \times \rho_{24}\, e^{\frac{1}{\hbar} p_1(u_4-u_2)} e^{\frac{\lambda_{24}}{2\hbar}(2u_3-2u_1+u_4-u_2)}\end{aligned}$$

Theorem. The 3D reflection equation with spectral parameters is valid:

$$\mathcal{R}_{457}\mathcal{K}_{4689}\mathcal{K}_{2379}\mathcal{R}_{258}\mathcal{R}_{178}\mathcal{K}_{1356}\mathcal{R}_{124} = \mathcal{R}_{124}\mathcal{K}_{1356}\mathcal{R}_{178}\mathcal{R}_{258}\mathcal{K}_{2379}\mathcal{K}_{4689}\mathcal{R}_{457}$$

where $\mathcal{R}_{ijk} = \mathcal{R}(\lambda_i, \lambda_j, \lambda_k)_{ijk}$ and $\mathcal{K}_{ijkl} = \mathcal{K}(\lambda_i, \lambda_j, \lambda_k, \lambda_l)_{ijkl}$.

4. Tetrahedron equality as duality

A representation of the q-Weyl algebra $e^{p_i}e^{u_j} = q^{2\delta_{ij}}e^{u_j}e^{p_i}$ on $\bigoplus_{m_1,m_2,m_3\in\mathbb{Z}^3}\mathbb{C}|m_1,m_2,m_3\rangle$

$$e^{p_i}|m_1,m_2,m_3\rangle = |m_1,m_2,m_3\rangle|_{m_i\to m_i-1}, \quad e^{u_i}|m_1,m_2,m_3\rangle = q^{2m_i}|m_1,m_2,m_3\rangle$$

Matrix elements :

$$R^{a,b,c}_{i,j,k} := \langle a,b,c|\mathcal{R}_{123}|i,j,k\rangle = \delta^{a+b}_{i+j}\delta^{b+c}_{j+k}\left(-\frac{\kappa_1}{\kappa_3}\right)^{b-k}\left(\frac{\kappa_2}{\kappa_3}\right)^{k-i}\frac{q^{(b-k)(i-k+1)}}{(q^2;q^2)_{b-k}}$$

Substitution of this into the tetrahedron equality

$$\begin{aligned}&\sum_{b_1,\dots,b_6\in\mathbb{Z}} R^{a_1,a_2,a_4}_{b_1,b_2,b_4}(\lambda_1,\lambda_2,\lambda_4)R^{b_1,a_3,a_5}_{c_1,b_3,b_5}(\lambda_1,\lambda_3,\lambda_5)R^{b_2,b_3,a_6}_{c_2,c_3,b_6}(\lambda_2,\lambda_3,\lambda_6)R^{b_4,b_5,b_6}_{c_4,c_5,c_6}(\lambda_4,\lambda_5,\lambda_6)\\ =&\sum_{b_1,\dots,b_6\in\mathbb{Z}} R^{a_4,a_5,a_6}_{b_4,b_5,b_6}(\lambda_4,\lambda_5,\lambda_6)R^{a_2,a_3,b_6}_{b_2,b_3,c_6}(\lambda_2,\lambda_3,\lambda_6)R^{a_1,b_3,b_5}_{b_1,c_3,c_5}(\lambda_1,\lambda_3,\lambda_5)R^{b_1,b_2,b_4}_{c_1,c_2,c_4}(\lambda_1,\lambda_2,\lambda_4),\end{aligned}$$

is distilled into the *duality* of q-series under the interchange $r \longleftrightarrow s$:

$$\frac{1}{(q^2)_{s+t}}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{n(n+1+2s)}}{(q^2)_n(q^2)_{t-n}(q^2)_{n+r}} = \frac{1}{(q^2)_{r+t}}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{n(n+1+2r)}}{(q^2)_n(q^2)_{t-n}(q^2)_{n+s}}$$

Possible connections with dualities in supersymmetric gauge theories (see Yagi arXiv:2405.02870)

A similar duality is present also in the *modular double* setting,
where the quantum dilogarithm is replaced by the "non-compact” counterpart (NCQD).

$$\Phi_b(u) = \exp\left(\frac{1}{4}\int_{\mathbb{R}+i0} \frac{e^{-2iuw}}{\sinh(wb)\sinh(w/b)}\frac{dw}{w}\right) \qquad q = e^{i\pi\mathfrak{b}^2}$$

The duality in that case emerges as an identity of integrals, which is also reproduced by a NCQD analogue of a classical Heine transformation.

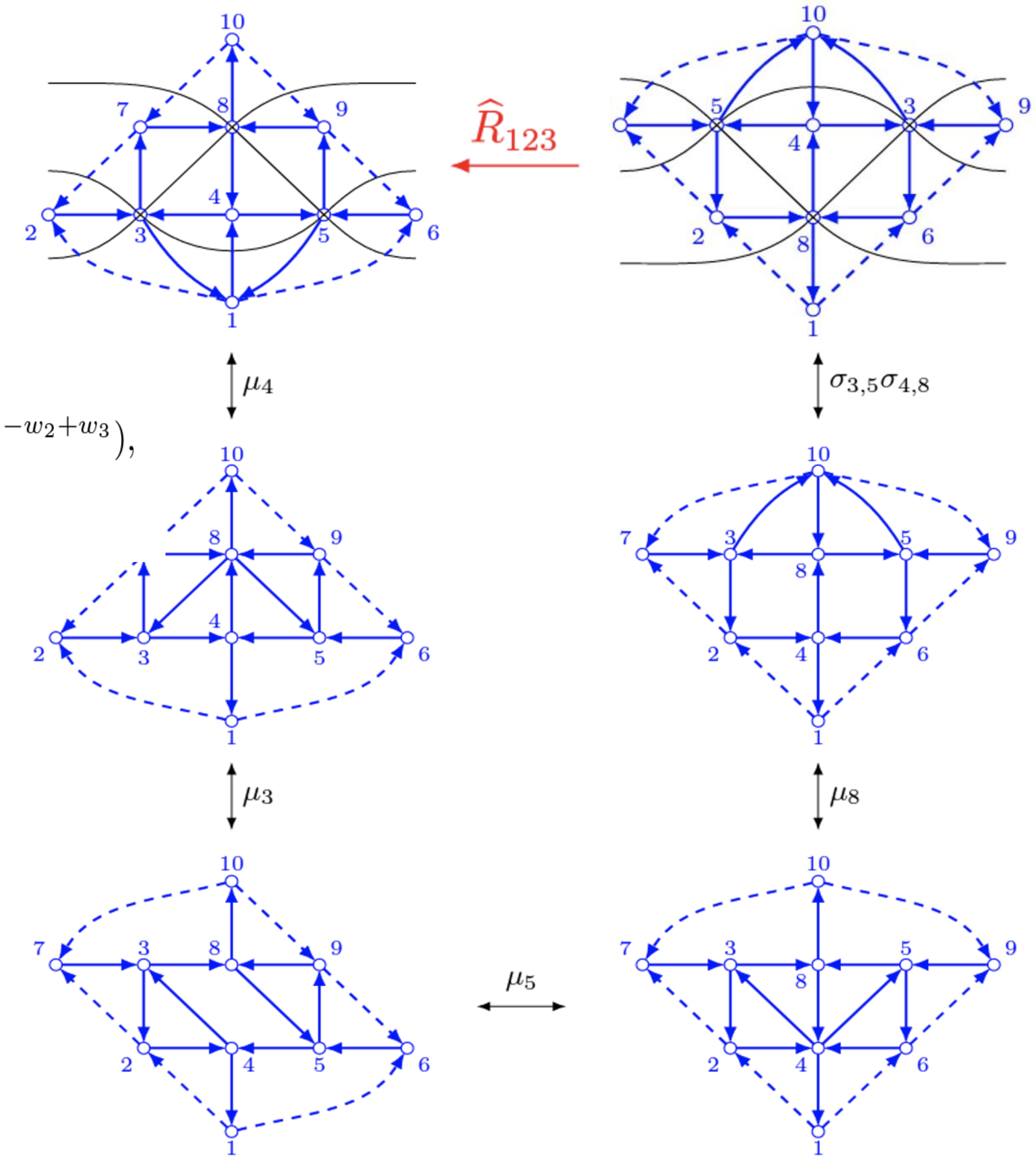
5. Outlook

3D R-matrix for Symmetric Butterfly (SB) quiver

(Inoue-K-Sun-Terashima-Yagi, 24)

Consists of 4 mutations.

$$R = \Psi_q\big(\mathrm{e}^{2C_7+u_1+u_3+w_1-w_2+w_3}\big)^{-1}\Psi_q\big(\mathrm{e}^{2C_5+u_1-u_3+w_1-w_2+w_3}\big)^{-1}$$
$$\times P\Psi_q\big(\mathrm{e}^{2C_2+2C_3-2C_6+2C_8+u_1-u_3+w_1-w_2+w_3}\big)\Psi_q\big(\mathrm{e}^{2C_2+2C_3+u_1+u_3+w_1-w_2+w_3}\big),$$
$$P = \mathrm{e}^{\frac{1}{\hbar}(u_3-u_2)w_1}\mathrm{e}^{\frac{1}{\hbar}\lambda_0(-w_1-w_2+w_3)}\mathrm{e}^{\frac{1}{\hbar}(\lambda_1 u_1+\lambda_2 u_2+\lambda_3 u_3)}\rho_{23},$$



Generalizes and unifies many known solutions as specializations of parameters in appropriate representations of Weyl algebras or their modular doubles.

- Kapranov-Voevodsky (94) q-oscillator representation
- Bazhanov-Mangazeev-Seregeev (09) coordinate representation
- K-Matsuike-Yoneyama (22) momentum representation
- Inoue-K-Terashima (23, this talk) specializing parameters

From BC (before the cluster alg.) to AD (after the dawn)

Quantum cluster algebras cover most of the prominent solutions of the tetrahedron equation.

Captured by quantum cluster algebra for **square quiver** [Inoue-K-Terashima 23]

$$\langle x|\mathcal{R}|x'\rangle \sim \delta(x_2+x_3-x_2'-x_3') \times \frac{\Phi_b(x_2-x_1\cdots)\Phi_b(x_2'-x_1'\cdots)}{\Phi_b(x_2'-x_1\cdots)\Phi_b(x_2-x_1'\cdots)}$$

"quantum 2+1 evolution model"

[Sergeev 98, 10]

"vertex-IRC" duality $\longleftrightarrow$

$\downarrow$ $q^N = 1$

$$R^{i_1i_2i_3}_{j_1j_2j_3} \sim \delta^{i_2+i_3}_{j_2+j_3}\frac{w_{p_1}(i_2-i_1)w_{p_2}(j_2-j_1)}{w_{p_3}(j_2-i_1)w_{p_4}(i_2-j_1)}$$

"vertex formulation of ZBB model"

[Sergeev-Mangazeev-Stroganov 95] $q^N = 1$

"vertex-IRC" duality $\longleftrightarrow$

Captured by quantum cluster algebra for **symmetric butterfly quiver** [I-K-Sun-T-Yagi 24]

Fock-Goncharov quiver (this talk) is the special case where only one of the four quantum dilogarithms Φ_b survives.

$$\langle x|R|x'\rangle \sim \frac{\Phi_b(z_1)\Phi_b(z_2)\Phi_b(z_3)\Phi_b(z_4)}{\Phi_b(z_3+z_4\cdots)}$$

$$(z_i = \text{linear form of } x_1,\dots,x_3')$$

moduar double of [K-Matsuike-Yoneyama 23]

$\updownarrow$ Fourier transform

$$\langle\sigma|R|\sigma'\rangle \sim \delta^{\sigma_1+\sigma_2}_{\sigma_1'+\sigma_2'}\delta^{\sigma_2+\sigma_3}_{\sigma_2'+\sigma_3'}\int dz\frac{e^{\cdots}\Phi_b(z+\frac{\sigma_1-\sigma_3\cdots}{2})\Phi_b(z+\frac{\sigma_3-\sigma_1\cdots}{2})}{\Phi_b(z+\frac{\sigma_1+\sigma_3\cdots}{2})\Phi_b(z-\frac{\sigma_1'+\sigma_3'\cdots}{2})}$$

"quantum geometry R"

[Bazhanov-Mangazeev-Sergeev 09]

$\downarrow$ $q^N = 1$

$$\langle n|R|n'\rangle \sim \delta^{n_1+n_2}_{n_1'+n_2'}\delta^{n_2+n_3}_{n_2'+n_3'}\sum_{n\in\mathbb{Z}_N}\frac{q^{\cdots}w_{p_1}(n+\frac{n_1-n_3\cdots}{2})w_{p_2}(n+\frac{n_3-n_1\cdots}{2})}{w_{p_3}(n+\frac{n_1+n_3\cdots}{2})w_{p_4}(n-\frac{n_1'+n_3'\cdots}{2})}$$

"Zamolodchikov-Bazhanov-Baxter (ZBB) model"

[Bazhanov-Baxter 92] $q^N = 1$